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Percolation in High Dimensions

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In 1990 Kesten [15] proved that the critical probability $p_c(\mathbb{Z}^n, \text{site})$ for site percolation in $\mathbb{Z}^n$ is at most $(1 + O((\log\log n)^2/\log n))/2n$. Together with the immediate lower bound of $1/(2n-1)$, this result shows that $p_c(\mathbb{Z}^n, \text{site}) = (1 + o(1))/2n$. Since the critical probability $p_c(\mathbb{Z}^n, \text{bond})$ for bond percolation in $\mathbb{Z}^n$ is no greater than $p_c(\mathbb{Z}^n, \text{site})$, and $p_c(\mathbb{Z}^n, \text{bond}) \geq 1/(2n-1)$ as well, we have that $p_c(\mathbb{Z}^n, \text{bond}) = (1 + o(1))/2n$ also holds (see also Gordon [9]). In a remarkable paper [12], in which Hara and Slade prove that the so-called triangle condition holds in certain percolation processes in $\mathbb{Z}^n$, it is shown that, in fact, $p_c(\mathbb{Z}^n, \text{bond}) = (1 + O(1/n))/2n$. In the same paper Hara and Slade also state in passing that their methods give $p_c(\mathbb{Z}^n, \text{site}) = (1 + O(1/n))/2n$.

The main aim of this note is to give a self-contained simple proof of the inequality $p_c(\mathbb{Z}^n, \text{site}) \leq (1 + n^{o(1)-1/3})/2n$. Our methods differ greatly from those of Kesten and of Hara and Slade; in particular, our proofs are entirely combinatorial. We then estimate $p_c(\mathbb{Z}^n, \text{bond})$ using a variant of our method, and give a simple proof of a result that is only slightly weaker than that of Hara and Slade, namely that $p_c(\mathbb{Z}^n, \text{bond}) \leq (1 + O((\log n)^2/n))/2n$.

1. INTRODUCTION

Let a graph G be given and let $0 \leq p \leq 1$. We denote by $\mathcal{G}_{\text{ind}}(G, p)$ the probability space of random induced subgraphs of G, where a vertex $v \in G$ is present in $G_p \in \mathcal{G}_{\text{ind}}(G, p)$ with probability p independently from all other vertices of G. Also, we denote by $\mathcal{G}(G, p)$ the space of random spanning subgraphs of G, where an edge e of G is present in $G_p \in \mathcal{G}(G, p)$ with probability p independently from all other edges of G. For graph-theoretical terminology not defined here, the reader is referred to [1].

Let K^n be the complete graph on n vertices. The asymptotic properties of the probability space $\mathcal{G}(K^n, p)$ of the *random graphs* $G_{n,p}$ have been investigated in great detail since the pioneering work of Erdős and Rényi in the 1960s, and by now a host of very interesting properties about $G_{n,p}$ are known (see, for instance, [1]). More recently, many authors have turned to the study of the spaces $\mathcal{G}(Q^n, p)$ and $\mathcal{G}_{\text{ind}}(Q^n, p)$, where Q^n is the *n-dimensional cube*, i.e. the graph the vertices of which are the 2^n subsets of a given n-element set and the edges of which join pairs of vertices the symmetric difference of which is a singleton. In [3] and [4] some very recent results concerning these spaces are presented, and the references in these papers give a good idea of what is known about 'phase transition' phenomena in $\mathcal{G}(Q^n, p)$ and $\mathcal{G}_{\text{ind}}(Q^n, p)$.

Now, in this note we shall be interested in random subgraphs of the integer lattice $\mathbb{Z}^n$, the infinite graph the vertices of which are the integer vectors of length n with two such vertices joined by an edge iff they are at distance 1. More specifically, we are concerned here with phase transition in the space $\mathcal{G}_{\text{ind}}(\mathbb{Z}^n, p)$ or, in other words, unoriented site percolation in $\mathbb{Z}^n$. We shall investigate the asymptotic behaviour of the critical probability for such percolation processes using methods that have been successfully applied in the study of $\mathcal{G}(Q^n, p)$ and $\mathcal{G}_{\text{ind}}(Q^n, p)$ (cf. [3, 4]).

Let us recall some basic facts about percolation in $\mathbb{Z}^n$. Let us denote by $\mathcal{G}^0_{\text{ind}}(\mathbb{Z}^n, p)$ the space of induced subgraphs $\mathbb{Z}^n_p$ of $\mathbb{Z}^n$ that contain the origin $0 \in \mathbb{Z}^n$ with the natural probability measure $\mathbb{P} = \mathbb{P}(\cdot \mid 0 \in \mathbb{Z}^n_p)$. Thus all the vertices of $\mathbb{Z}^n$ are present in $\mathbb{Z}^n_p \in \mathcal{G}^0_{\text{ind}}(\mathbb{Z}^n, p)$ independently with probability p, except for 0, which is always present. For a graph $G \subset \mathbb{Z}^n$ and a vertex $x \in G$, let us denote the component of x in G by

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$C_x = C_x(G)$. A basic problem in percolation theory is to determine the probability $\theta(p) = \theta^n(p)$ (resp. $\theta_{\mathrm{ind}}(p) = \theta^n_{\mathrm{ind}}(p)$) that $C_0 = C_0(\mathbb{Z}^n_p)$ is infinite, where $\mathbb{Z}^n_p \in \mathcal{G}(\mathbb{Z}^n, p)$ (resp. $\mathbb{Z}^n_p \in \mathcal{G}^0_{\mathrm{ind}}(\mathbb{Z}^n, p)$).

The *critical probability for bond percolation* in $\mathbb{Z}^n$ is $p_c(\mathbb{Z}^n, \text{bond}) = \sup\{p: \theta^n(p) = 0\}$ and *for site percolation* is $p_c(\mathbb{Z}^n, \text{site}) = \sup\{p: \theta^n_{\mathrm{ind}}(p) = 0\}$. It is known that these critical probabilities are strictly between 0 and 1 if $n \geq 2$. Let us mention that the only critical probability that is known explicitly is $p_c(\mathbb{Z}^2, \text{bond})$, which equals 1/2. This is a celebrated result of Kesten [14] (see also [10, Chapter 9]) and its proof is remarkably difficult. Our object here is to study the asymptotic behaviour of the critical probabilities as $n \to \infty$.

The two percolation processes described above are sometimes referred to as *unoriented* percolation processes. In an *oriented* process we are interested in the existence of an infinite path $x_0x_1x_2 \cdots$ in $\mathbb{Z}^n$ such that $x_0 = 0 \in \mathbb{Z}^n$ and x_i is further away from $0 \in \mathbb{Z}^n$ than x_{i-1} $(i \geq 1)$. In [8] Cox and Durret proved very sharp bounds for the critical probability of oriented bond percolation. Their result is that

$$\frac{1}{n} + \frac{1}{2n^3} + o\left(\frac{1}{n^3}\right) \leq p_{c,\mathrm{or}}(\mathbb{Z}^n, \text{bond}) \leq \frac{1}{n} + \frac{1}{n^3} + O\left(\frac{1}{n^4}\right). \tag{1}$$

Following Kesten, Cox and Durret proved that $p_{c,\mathrm{or}}(\mathbb{Z}^n, \text{bond})$ is bounded from above by the probability that a certain pair of random walks share an edge. This latter probability is then shown to be at most the right-hand side of (1).

Now, the unoriented case seems to be a great deal more complicated, although Kesten's approach basically still works. Elaborating his technique, Kesten [15] gave an ingenious proof of the inequality

$$p_c(\mathbb{Z}^n, \text{site}) \leq \frac{1}{2n} + O\left(\frac{(\log \log n)^2}{n \log n}\right).$$

(See also Gordon [9], where it is proved that $p_c(\mathbb{Z}^n, \text{bond}) = (1 + o(1))/2n$.) The main aim of this paper is to improve this result by showing that, in fact, this critical probability is bounded from above by

$$\frac{1}{2n} + O\left(\frac{1}{n^{4/3 - o(1)}}\right).$$

Our proof is entirely different from Kesten's, and is more combinatorial than analytical in nature. We prove our bound through a short sequence of combinatorial lemmas most of which are very natural if we keep in mind some basic results about branching processes. Using these lemmas, we can show our upper bound by reducing our site percolation in $\mathbb{Z}^n$ to oriented site percolation in $\mathbb{Z}^2$.

The situation for bond percolation in high dimensions is, in many respects, much more satisfactory. In [12] Hara and Slade proved the remarkable result that $p_c(\mathbb{Z}^n, \text{bond}) = (1 + O(1/n))/2n$. To be precise, they gave a rather long and involved proof that a deep hypothesis known as the triangle condition holds for certain $\mathbb{Z}^n$ percolation processes, and deduced that $p_c(\mathbb{Z}^n, \text{bond}) = (1 + O(1/n))/2n$ from this result. Furthermore, Hara and Slade [12] stated in passing that $p_c(\mathbb{Z}^n, \text{site}) = (1 + O(1/n))/2n$; unfortunately, they gave no indication as to the actual proof. It would be most desirable to give a simple and insightful proof of this result; our methods might be a first step towards this aim.

Finally, let us give a brief outline of the contents of this note. In Section 2 we give a preliminary result (Corollary 4), which tells us that as far as local properties are concerned there is not much difference between percolation in $\mathbb{Z}^n$ and percolation in an infinite $2n$-regular tree. In Section 3 we improve this observation and prove Lemma 5, which is crucial in the proof of Lemma 7 given in the next section. We present our main result in Section 5. We shall see that its proof is a simple application of Lemma 7. The final section is about bond percolation: we indicate how our methods can be used to give a simple proof of an upper bound for the critical probability $p_c(\mathbb{Z}^n, \text{bond})$ that is not much worse than the bound due to Hara and Slade.

2. Preliminaries

The main aim of this section is to show that if the origin $0 \in \mathbb{Z}^n$ belongs to $\mathbb{Z}_p^n \in \mathcal{G}_{\text{ind}}(\mathbb{Z}^n, p)$, then with high probability its component has moderately large order, say at least $n^{2/3}$. In fact, we shall show that this happens with probability at least about 2ε if $p = (1+\varepsilon)/2n$ and $\varepsilon = \varepsilon(n) = o(n)$ is not too small. We shall see that this bound for this probability is best possible by considering certain branching processes.

Let us now define a Galton–Watson branching process that will be needed later. Let $n \geqslant 2$ and $0 \leqslant p = p(n) \leqslant 1$ be given. We shall make use of the branching process $\Pi_n^-(p) = (Z_t)_{t=0}^{\infty}$ that can be described as follows. In $\Pi_n^-(p)$ we start with one particle that generates offspring according to the binomial distribution $\mathrm{Bi}(n, p)$ with parameters n and p, and all other particles generate offspring according to $\mathrm{Bi}(n-1, p)$. Note that the critical probability for this branching process is $1/(n-1)$. Let the probability that $\Pi_n^-(p)$ does not die out be $\pi_n^-(p)$. The following lemma can be checked easily. (We refer the reader to the monograph by Harris [13] for an introduction to branching processes, and for a brief discussion on results such as Lemma 1 below, the reader is referred to [3, Section 2].)

Lemma 1. *Let $p = (1+\varepsilon)/n$, where $\varepsilon = \varepsilon(n) = o(n)$ and $\varepsilon n \to \infty$ as $n \to \infty$. Then $\pi_n^-(p) = (2 + o(1))\varepsilon$.*

It is not difficult to see that we may generate the component C_0 of $0 \in \mathbb{Z}^n$ in $\mathbb{Z}_p^n \in \mathcal{G}_{\text{ind}}(\mathbb{Z}^n, p)$, conditional on $0 \in \mathbb{Z}_p^n$, with a process resembling the branching process $\Pi_{2n}^-(p)$. We shall make this remark precise in what follows. Our aim is to show that if $0 \in \mathbb{Z}^n$ belongs to $\mathbb{Z}_p^n$ then it lies in a component C_0 of order at least about $n^{2/3}$ with probability about $\pi_{2n}^-(p)$. One difficulty is that we have cycles in $\mathbb{Z}^n$ and hence there is not as much independence in the generation of C_0 as there is in the generation of the branching process $\Pi_{2n}^-(p)$. To overcome this difficulty we shall need the lemma below.

Lemma 2. *Let $v \in \mathbb{Z}^n$ be fixed and $k \geqslant 2$. The number of cycles of length $2k$ in $\mathbb{Z}^n$ containing v is not greater than*

$$\binom{2k}{k} k!\, n^k.$$

Proof. Let us just sketch the proof. The $2k$-cycles C containing a fixed vertex v can be counted as follows. First, we see that exactly k of the 'steps' in C are 'positive' (i.e.

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begin
  H^p(v) := v;
  insert v in a queue Q;
  repeat
    w := first element in Q;
    delete w from Q;
    let w_1, ..., w_m be the neighbours of w in Z^n
         not contained in H^p(v), in their natural order;
         {N.B. m = 2n the first time and m = 2n - 1 every other time}
    for i = 1, ..., m do
      with probability p do begin
         insert w_i at the back of Q;
         add w_i to H^p(v);
         if H^p(v) is not acyclic or |H^p(v)| = n_0
              then output H^p(v) and halt
         end
  until Q is empty;
  output H^p(v)
end
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FIGURE 1. Algorithm I.

of the form e_i for some i) and the other k are 'negative' (i.e. of the form $-e_i$ for some i). The factor $\binom{2k}{k}$ accounts for the order in which the positive and negative steps come in the cycle. Now, the choice of the k positive steps gives us a factor of n^k, but the k negative steps give us a factor of $k!$. This last statement follows from the fact that the negative steps must be just a permutation of the reversals of the k positive ones. □

Let us now consider a probabilistic algorithm, Algorithm I, that generates induced subgraphs of $\mathbb{Z}^n$ containing a fixed vertex $v \in \mathbb{Z}^n$. Let $0<\delta<1/3$ be fixed. Let $0 \leqslant p = p(n) \leqslant 1$ and write $n_0 = \lfloor n^{\delta+2/3}/\log n \rfloor$. For convenience, let us consider the unit vectors $\{\pm e_i: 1 \leqslant i \leqslant n\}$ totally ordered so that in increasing order they are $e_1, \ldots, e_n, -e_1, \ldots, -e_n$. Given $w \in \mathbb{Z}^n$ the $2n$ edges of $\mathbb{Z}^n$ incident to w may then be considered ordered in a canonical way, and so can the $2n$ neighbours of w. Since in Algorithm I we generate induced subgraphs of $\mathbb{Z}^n$, it is enough for us to specify its vertex set. Our algorithm is given in Figure 1. Note that this algorithm simulates $\Pi_{2n}^{-}(p)$ up to a certain stage and it outputs the result as $H^p(v)$.

Note that a vertex of $\mathbb{Z}^n$ may be considered up to $n_0 - 1$ times in our algorithm, but at most one vertex of $H^p(v)$ has been considered more than once. More precisely, we see that $H^p(v)$ contains a vertex that was considered more than once only if $H^p(v)$ has a cycle, and in fact if $H^p(v)$ does contain a vertex w examined more than once, then w was the last vertex put into $H^p(v)$, and any cycle contained in $H^p(v)$ goes through w. Note also that if H_0 is an induced tree of $\mathbb{Z}^n$ that has order not greater than n_0 and $v \in H_0$, then there is only one way in which our algorithm can generate H_0. We summarize the relevant properties of Algorithm I in the lemma below.

LEMMA 3. *Let* $0<\delta<1/3$ *be fixed. Let* $0<p=p(n)\leqslant 1/n$ *be given and let* $n_0 = \lfloor n^{\delta+2/3}/\log n \rfloor$. *Let* v *be any vertex of* $\mathbb{Z}^n$. *Then*
(i) *the probability that* $H^p(v)$ *is acyclic and has order less than* n_0 *is less than* $1-\pi_{2n}^{-}(p)$,
(ii) *the probability that* $H^p(v)$ *contains a cycle is* $O(n^{\delta-1/3}/\log n)$,

(iii) *the probability that $H^p(v)$ is acyclic and has order n_0 is at least*

$$\pi_n^-(p) + O(n^{\delta-1/3}/\log n).$$

PROOF. (i) Recall that our algorithm simulates the branching process $\Pi_{2n}^-(p)$ up to a certain stage, and it only generates an $H^p(v)$ that is acyclic and has fewer than n_0 vertices if the corresponding branching process dies out, which happens with probability $1-\pi_{2n}^-(p)$.

(ii) Let $2 \leq k \leq n_0/2$. Lemma 2 above tells us that the number of cycles of length $2k$ of $\mathbb{Z}^n$ that contain a given fixed vertex of $\mathbb{Z}$ is bounded from above by

$$\binom{2k}{k} k!\, n^k = O\left[\left(\frac{4kn}{e}\right)^k\right].$$

Let us assume that our algorithm has generated an $H^p(v)$ that contains a cycle $C = C^{2k}$ of length $2k$, and let the last vertex added to $H^p(v)$ be $w \in C$. Let us first consider the case in which C does not contain v. In this case, each vertex of C has been examined by our algorithm exactly once, except possibly for w, which might have been examined up to $n_0 - 1$ times. Hence the probability that there is such a C contained in $H^p(v)$ is at most

$$2kn_0\binom{2k}{k} k!\, n^k \cdot n^{-(2k-1)}(n_0/n) = O\big[k(4k/e)^k n^{2\delta+4/3-k}\big].$$

Summing these probabilities over k, we see that the probability that our algorithm has generated an $H^p(v)$ containing a cycle that does not pass through v is at most $O(n^{2\delta-2/3})$.

Let us now consider the case in which we have generated an $H^p(v)$ that contains a cycle $C = C^{2k}$ of length $2k$ that goes through v. In this case, each vertex of C has been examined by our algorithm exactly once; except for v, which has not been examined at all, and possibly w, which might have been examined up to $n_0 - 1$ times. Hence, the probability that such a C exists is bounded from above by

$$2k\binom{2k}{k} k!\, n^k \cdot n^{-(2k-2)}(n_0/n) = O\big[k(4k/e)^k n^{\delta+5/3-k}/\log n\big].$$

Summing over k we see that the probability that our algorithm has generated an $H^p(v)$ containing a cycle that does pass through v is at most $O(n^{\delta-1/3}/\log n)$, and this completes the proof of (ii).

(iii) This follows from (i) and (ii). □

We are now ready to prove the main result of this section.

COROLLARY 4. *Let $0 < \delta < 1/3$ be fixed. Let $0 < p = p(n) \leq 1/n$ be given and let $n_0 = \lfloor n^{\delta+2/3}/\log n \rfloor$. Let v be any vertex of $\mathbb{Z}^n$. Then*

$$\mathbb{P}(|C_v| \geq n_0 \mid v \in \mathbb{Z}_p^n) \geq \pi_{2n}^-(p) + O(n^{\delta-1/3}/\log n),$$

where $C_v = C_v(\mathbb{Z}_p^n)$ is the component of $\mathbb{Z}_p^n \in \mathcal{G}_{\mathrm{ind}}(\mathbb{Z}^n, p)$ containing v.

PROOF. Assume that $v \in \mathbb{Z}_p^n$. Let $W \subset \mathbb{Z}^n$ be the first $\min\{|C_v|, n_0\}$ vertices reached by the canonical breadth-first search run on C_v starting at v. Let the subgraph $\mathbb{Z}^n[W]$ of $\mathbb{Z}^n$ induced by W be denoted by H_v. Clearly, $|C_v| \geq n_0$ iff $|H_v| \geq n_0$. We claim that the probability that H_v is acyclic and has at least n_0 vertices is at least $\pi_{2n}^-(p) + O(n^{\delta-1/3}/\log n)$. Clearly, this claim implies our result.

Let $H_0 \subset \mathbb{Z}^n$ be an acyclic induced subgraph of $\mathbb{Z}^n$ that contains v and has order n_0. Recall that there is only one way in which our probabilistic algorithm can generate H_0 as its output $H^p(v)$. It is easily seen that there is an integer $L = L(H_0) \geqslant 0$ such that

$$\mathbb{P}(H^p(v) = H_0) = \mathbb{P}(H_v = H_0)(1-p)^L \leqslant \mathbb{P}(H_v = H_0),$$

and hence our claim follows from Lemma 3(iii) by summing over all possible H_0. □

3. The First Main Lemma

In this section we aim to prove a strengthening of Corollary 4: we shall prove that n_0 in the statement of that corollary may be replaced by any fixed polynomial of n.

Let $v = (v_i)_1^n \in \mathbb{Z}^n$ and $S \subset [n] = \{1, \ldots, n\}$ be given. We shall denote by $v|S$ the vector $(v_i)_{i \in S}$, and we shall call a set $Z \subset \mathbb{Z}^n$ of the form $Z = \{z \in \mathbb{Z}^n\colon\ z|S = v|S\}$ a *flat*. If $|S| = k$ we say that Z above has *dimension* $n-k$.

Lemma 5. *Let $0 < \delta < 1/3$ be fixed and let an integer $C \geqslant 1$ be given. Then there is a constant $\alpha = \alpha(C, \delta) > 0$ such that for any $v \in \mathbb{Z}^n$ the following holds. If $p = (1+\varepsilon)/2n$, where $n^{\delta - 1/3} \leqslant \varepsilon = \varepsilon(n) = o(n)$, then*

$$\mathbb{P}\{|C_v| \geqslant \alpha(n^{2\delta}/\log n)^C \mid v \in \mathbb{Z}_p^n\} \geqslant (2 + o(1))\varepsilon,$$

where $C_v = C_v(\mathbb{Z}_p^n)$ is the component of $\mathbb{Z}_p^n \in \mathcal{G}_{\mathrm{ind}}(\mathbb{Z}^n, p)$ that contains v.

Proof. We shall prove our lemma by induction on C. If $C = 1$ then the result holds by Lemma 1 and Corollary 4, and hence we may proceed to the induction step. Let $C \geqslant 2$ be fixed, and assume that our result holds for $C-1$.

For a vertex $v \in \mathbb{Z}^n$, let us denote by $\mathcal{G}_{\mathrm{ind}}^v(\mathbb{Z}^n, p) \subset \mathcal{G}_{\mathrm{ind}}(\mathbb{Z}^n, p)$ the set of the $\mathbb{Z}_p^n \in \mathcal{G}_{\mathrm{ind}}(\mathbb{Z}^n, p)$ such that $v \in \mathbb{Z}_p^n$. Let $v \in \mathbb{Z}^n$ be fixed. We shall generate a random $\mathbb{Z}_p^n \in \mathcal{G}_{\mathrm{ind}}^v(\mathbb{Z}^n, p)$ in an indirect way, and then prove that the component $C_v = C_v(\mathbb{Z}_p^n)$ of $\mathbb{Z}_p^n$ containing v is large with high probability. Indeed, to take advantage of independence we shall generate $\mathbb{Z}_p^n \in \mathcal{G}_{\mathrm{ind}}^v(\mathbb{Z}^n, p)$ in two rounds. More precisely, let $p_2 = n^{-4/3}$ and set $p_1 = (p - p_2)/(1-p_2)$, so that $p = p_1 + p_2 - p_1 p_2$. A way of generating a random $\mathbb{Z}_p^n \in \mathcal{G}_{\mathrm{ind}}^v(\mathbb{Z}^n, p)$ is to pick $G_i \in \mathcal{G}_{\mathrm{ind}}^v(\mathbb{Z}^n, p_i)$ $(i = 1, 2)$ independently, and then let $\mathbb{Z}_p^n$ be such that $V(\mathbb{Z}_p^n) = V(G_1) \cup V(G_2)$.

Let $n_0 = \lfloor n^{\delta + 2/3}/\log n \rfloor$. Let us consider Algorithm II, given in Figure 2. This is a probabilistic algorithm that generates a random connected induced subgraph $J^{p_1}(v)$ of $\mathbb{Z}^n$ containing v. Since we are generating an induced subgraph, it suffices to specify the vertices of $J^{p_1}(v)$. Algorithm II builds $J^{p_1}(v)$ by selecting its vertices one by one, in a certain random fashion.

We shall generate our G_1 by first executing Algorithm II and then deciding whether or not the vertices in $\mathbb{Z}^n \setminus A$ should be in G_1, where A is the set of vertices examined by our algorithm. We want to show that we succeed in generating a $\mathbb{Z}_p^n$ with $C_v(\mathbb{Z}_p^n)$ of large order with high probability. Let us assume that we have run Algorithm II and it has generated $J_0 = J^{p_1}(v)$ of order n_0. Moreover, let us denote by A_0 the set of the vertices of $\mathbb{Z}^n$ that have been examined by our algorithm. Note that

$$\mathbb{P}(|J_0| = n_0) = \mathbb{P}(|C_v(G_1)| \geqslant n_0 \mid v \in G_1),$$

and hence by Corollary 4 we have that $J_0 = J^{p_1}(v)$ has n_0 vertices with probability at least $\pi_{2n}^-(p_1) + O(n^{\delta - 1/3}/\log n)$, which equals $(1 + o(1))\pi_{2n}^-(p_1) = (2 + o(1))\varepsilon$ since $p_1 = (1 + (1 + o(1))\varepsilon)/2n$.

Let $V(J_0) = \{v_i\colon 1 \leqslant i \leqslant n_0\}$. By a variant of a simple result of Bondy [5] (see also [2], Chapter 2), there is a set $I \subset [n]$ with $|I| = n_0 - 1$ such that the $v_i|I$ $(1 \leqslant i \leqslant n_0)$ are all

begin
  $J^{p_1}(v) := v$; $A := \{v\}$;
  insert v into a queue Q;
  repeat
    $w :=$ first element of Q;
    delete w from Q;
    let $w_1, \ldots, w_m$ be the neighbours of w in $\mathbb{Z}^n$
      not contained in $J^{p_1}(v)$, in their natural order;
    for $i = 1, \ldots, m$ **do begin**
      $A := A \cup \{w_i\}$;
      with probability p_1 **do begin**
        add w_i to $J^{p_1}(v)$;
        insert w_i at the back of Q;
        if $|J^{p_1}(v)| = n_0$ **then** output $J^{p_1}(v)$ and halt
        end
      end
  until Q is empty;
  output $J^{p_1}(v)$
end

FIGURE 2. Algorithm II.

distinct. Let us define n_0 pairwise disjoint flats of $\mathbb{Z}^n$, each containing exactly one vertex of J_0. We set

$$Z_i = \{w \in \mathbb{Z}^n\colon\ w|I = v_i|I\}$$

for $1 \leqslant i \leqslant n_0$. Note that the dimension of the Z_i is $m_0 = n - n_0 + 1 = (1 + o(1))n$. Let us denote by V_0 the set of the vertices of $\bigcup Z_i$ that are adjacent to some vertex of J_0. Note that V_0 is the union of the sets V_i given by

$$V_i = V_0 \cap Z_i = \{w \in Z_i\colon w \text{ is adjacent to } v_i\},$$

$i = 1, \ldots, n_0$. Let us also note that the vertices in A_0 examined by our algorithm are such that $A_0 \cap Z_i \subset V_i$. For all i we clearly have $|V_i| = 2m_0$ and hence $|V_0| = 2n_0m_0$, since the $V_i = V_0 \cap Z_i$ are pairwise disjoint. Let us now randomly choose a subset V_0' of V_0 by letting, for all $w \in V_0$,

$$\mathbb{P}(w \in V_0') = p_2 = n^{-4/3},$$

all of these events being mutually independent. Also, let us set $V_i' = V_i \cap V_0'$. Note that $\mathbb{E}(|V_0'|) = 2n_0m_0n^{-4/3} = (2 + o(1))n^{\delta+1/3}/\log n$ and $\mathbb{E}(|V_i'|) = 2m_0n^{-4/3} = (2 + o(1))n^{-1/3}$. Rather crudely, we see that with probability $1 - o(\varepsilon)$ we have that

$$|V_0'| \geqslant n^{\delta+1/3}/\log n \tag{2}$$

and

$$n_i = |V_i'| \leqslant n_0 \tag{3}$$

for all $i = 1, \ldots, n_0$. Let us assume that the set V_0' that we have randomly chosen does satisfy (2) and (3) above. Let us now define a collection of $\sum n_i$ pairwise disjoint flats Z_{ij} of dimension $m_1 = m_0 - n_0 \geqslant n(1 - 2n^{\delta-1/3}/\log n)$. Let us write

$$V_i' = \{v_{ij}\colon 1 \leqslant j \leqslant n_i\},$$

for all i. From the neighbours of v_i in Z_i, let us arbitrarily choose n_0 vertices to form a set $W_i \subset V_i \subset Z_i$, so that $V_i' \subset W_i$. Set $I_i = I \cup \bigcup_{w \in W_i} \operatorname{supp}(v_i - w)$. Finally, we define

$$Z_{ij} = \{w \in Z_i\colon w|I_i = v_{ij}|I_i\},$$

for all i and j. Let us now randomly pick $Z_{ij}^p \in \mathcal{G}_{\mathrm{ind}}^{v_{ij}}(Z_{ij}, p)$. Let us denote by C_{ij} the component of Z_{ij}^p that contains v_{ij}. By the induction hypothesis, there is a constant $\alpha' = \alpha(C-1, \delta) > 0$ such that

$$\mathbb{P}\{|C_{ij}| \geq \alpha'(m_1^{2\delta}/\log m_1)^{C-1}\} \geq (2+o(1))\varepsilon.$$

Note that $(m_1^{2\delta}/\log m_1)^{C-1} \geq \beta(n^{2\delta}/\log n)^{C-1}$, where $\beta = \beta_{C,\delta} > 0$ depends only on C and δ. Let the r.v. X_{ij} be the indicator function of the event $|C_{ij}| \geq \alpha'\beta(n^{2\delta}/\log n)^{C-1}$, and set $X = \sum_{i,j} X_{ij}$. Recalling that $|V_0'| = \sum n_i \geq n^{\delta+1/3}/\log n$, we have that

$$\mathbb{E}(X) \geq (2+o(1))\varepsilon n^{\delta+1/3}/\log n \geq (2+o(1))n^{2\delta}/\log n,$$

and hence we have that $X \geq n^{2\delta}/\log n$ with probability $1-o(\varepsilon)$. Thus we have that

$$|C_v| = |C_v(\mathbb{Z}_p^n)| \geq \alpha'\beta(n^{2\delta}/\log n)^C$$

with probability $1-o(\varepsilon)$, and therefore we can take $\alpha(C,\delta) = \alpha(C-1,\delta)\beta_{C,\delta}$. This completes the induction step and hence the proof of our lemma. □

In the next section we shall use the following immediate corollary of the above lemma.

Corollary 6. *Let $0 < \delta < 1/3$ and $C \geq 1$ be fixed. Let $p = (1+\varepsilon)/2n$, where $n^{\delta-1/3} \leq \varepsilon = \varepsilon(n) = o(1)$. Then*

$$\mathbb{P}(|C_v| \geq (2n)^C \mid v \in \mathbb{Z}_p^n) \geq (2+o(1))\varepsilon,$$

where $C_v = C_v(\mathbb{Z}_p^n)$ is the component of $v \in \mathbb{Z}^n$ in $\mathbb{Z}_p^n \in \mathcal{G}_{\mathrm{ind}}(\mathbb{Z}^n, p)$.

4. The Second Main Lemma

Let $0 < \delta < 1/3$ be fixed. Let a collection of vertices $S \subset \mathbb{Z}^n$ be given and let us select a random $\mathbb{Z}_p^n \in \mathcal{G}_{\mathrm{ind}}(\mathbb{Z}^n, p)$, where $p = (1+\varepsilon)/2n$ and $2n^{\delta-1/3} \leq \varepsilon = \varepsilon(n) = o(n)$. Let $Z \subset \mathbb{Z}^n$ be the induced subgraph of $\mathbb{Z}^n$ with vertex set $S \cup \mathbb{Z}_p^n$. Note that Z is just a random induced subgraph of $\mathbb{Z}^n$ that contains S. What is the probability that in Z a vertex x of S lies in a component $C_x = C_x(Z)$ of 'large' order, say such that $|C_x| \geq n^{10}$? In the lemma below we show that this probability is $1-o(1)$ if S has at least n^{10} vertices. Note that the events $|C_x| \geq n^{10}$ $(x \in S)$ are *not* mutually independent. In the proof of Lemma 7 we shall consider more restricted events, and these will be independent. For a pair $(a, b) \in \mathbb{Z}_+^2$, let us write

$$H_{a,b} = H_{a,b}^n = \{(x_i)_1^{n+2} \colon x_1 = a, x_2 = b\} \subset \mathbb{Z}^{n+2}.$$

Thus $H_{a,b}$ is an n-dimensional flat of $\mathbb{Z}^{n+2}$.

Lemma 7. *Let $0 < \delta < 1/3$ and $C \geq 10$ be fixed. Let $p = (1+\varepsilon)/2n$ and $\varepsilon = \varepsilon(n) = 2n^{\delta-1/3}$. Let a set $S \subset H_{a,b}$ with $|S| \geq n^C$ be given. For $x \in H_{a,b}^p \in \mathcal{G}_{\mathrm{ind}}(H_{a,b}, p)$, let $C_x = C_x(H_{a,b}^p)$ be the component of x in $H_{a,b}^p$. Then*

$$\mathbb{P}(x \in H_{a,b}^p \text{ and } |C_x| \geq n^C \text{ for some } x \in S) = 1 - o(1).$$

Proof. Let us generate $H_{a,b}^p$ in two rounds. Let $p_0 = n^{-2}$ and p_1 be so that $p = p_0 + p_1 - p_0 p_1$. Note that if $p_1 = (1+\varepsilon_1)/2n$, then $\varepsilon_1 = (1+o(1))\varepsilon$.

For brevity, let us say that a component of a graph is *large* if it has order at least n^C. Pick $H_1 \in \mathcal{G}_{\mathrm{ind}}(H_{a,b}, p_1)$. Let us say that a vertex $x \in H_{a,b}$ is *good* if it is at distance at most 3 from a large component of H_1, and let us otherwise say that it is *bad*.

Claim. *With probability* $1-o(1)$ *at least* $n^C/2$ *vertices in* S *are good.*

Let us prove the claim. Let us estimate the probability that a fixed vertex $x \in H_{a,b}$ is bad. We may clearly assume that $a=b=0$ and that $x=0 \in H_{0,0}=\mathbb{Z}^n$. Let $k=n_0=\lfloor n^{\delta+2/3}/\log n \rfloor$ and set $m=n-k$ and $\ell=\binom{k+2}{3}$. Let $v_1, v_2, \ldots \in \mathbb{Z}^k$ be an enumeration of the integer vectors in $\mathbb{Z}^k$ at distance 3 from $0 \in \mathbb{Z}^k$. Note that, very crudely, there are at least ℓ such vectors. Let w_i be the image of v_i under the natural embedding $\mathbb{Z}^k \subset \mathbb{Z}^n$. Clearly, the distance between w_i and $0 \in \mathbb{Z}^n$ is 3. For $1 \leqslant i \leqslant \ell$, let W_i be the m-dimensional flat $\{w \in \mathbb{Z}^n\colon w|[k]=v_i\}$. Note that $w_i \in W_i$.

Let us note that, rather crudely, $(2m)^C \geqslant n^C$ and

$$p_1=\frac{1+\varepsilon_1}{2n}=\frac{1+\varepsilon_1}{2m}\left(1-\frac{k}{n}\right)\geqslant\frac{1+\varepsilon/2}{2m}.$$

Now we apply Corollary 6 to each pair (w_i, W_i). Then we see that $w_i \in W_i$ belongs to $W_i^p \in \mathcal{G}_{\text{ind}}(W_i, p)$ and lies in a component of order a least $(2m)^C \geqslant n^C$ (and hence is large) with probability at least $(2+o(1))\varepsilon/4m$. Thus we see that the probability that $0 \in \mathbb{Z}^n$ is bad is at most

$$(1-\varepsilon/3n)^\ell \leqslant \exp(-\varepsilon\ell/3n)=o(1).$$

Hence the probability that at least $|S|/2 \geqslant n^C/2$ vertices in S are bad is $o(1)$, by Markov's inequality. The proof of the claim is complete.

With the aid of our claim the proof of the lemma is simple. Let us assume that $H_1 \in \mathcal{G}_{\text{ind}}(H_{a,b}, p_1)$ is so that at least $n^C/2$ of the vertices in S are good. Let us note that then trivially there are at least $u=(n^C/2)/4n^3 \geqslant n^{C-3}/8$ pairwise vertex-disjoint paths from S to large components of H_1. Hence the probability that, when we select $H_0 \in \mathcal{G}_{\text{ind}}(H_{a,b}, p_0)$ and let $V(H_{a,b}^p)=V(H_0) \cup V(H_1)$, no vertex of S is joined up to a large component of H_1 is bounded from above by

$$(1-p_0^3)^u \leqslant \exp\{-n^{C-9}/8\}=o(1),$$

as required. □

5. The Main Result

We are now ready to prove the main result of this note. With the aid of Lemma 7 we shall see that Theorem 8 below can be deduced from a simple result on two-dimensional site percolation processes, which we shall now describe.

Let $0 \leqslant p_0 \leqslant 1$ and $0 \leqslant p_1 \leqslant 1$ be given. Let us generate random induced subgraphs of $\mathbb{Z}_+^2$ (the first quadrant of $\mathbb{Z}^2$) in the following way. We first delete $0 \in \mathbb{Z}_+^2$ with probability $1-p_0$. With probability $1-p_0$ our subgraph is empty. Suppose we did *not* delete $0 \in \mathbb{Z}^2$. Now we delete vertices of $\mathbb{Z}_+^2 \backslash \{0\}$ independently with probability $1-p_1$. Let the component of $0 \in \mathbb{Z}_+^2$ be D_0. Orient the edges in D_0 from left to right and upwards, i.e. so that every edge points away from $0 \in \mathbb{Z}_+^2$. Finally, let D_1 be the set of vertices in D_0 that can be reached from $0 \in \mathbb{Z}_+^2$ by a directed path in D_0. Let us denote by $\vec{\mathbb{Z}}_{p_0,p_1}^2$ the graph induced by D_1 in $\mathbb{Z}_+^2$. Using a Peierls type argument (see, for example, [10, p. 16]), one can easily check that, for any $\eta>0$,

$$\mathbb{P}(|\vec{\mathbb{Z}}_{p_0,p_1}^2|=\infty) \geqslant (1-\eta)p_0$$

if $p_1 \geqslant 1-\nu$, where $\nu=\nu(\eta)>0$ depends only on η.

We shall use the two-dimensional percolation process described above as follows.

We aim at estimating the probability $P_\infty = \mathbb{P}(|C_0| = \infty \mid 0 \in \mathbb{Z}_p^n)$, where $C_0 = C_0(\mathbb{Z}_p^n)$ is the component of $0 \in \mathbb{Z}_p^n \in \mathscr{G}_{\text{ind}}(\mathbb{Z}^n, p)$. In order to carry out this estimation, we shall run a probabilistic algorithm, Algorithm III, that generates C_0 at the same time that it generates $\vec{\mathbb{Z}}^2_{p_0,p_1}$ for certain probabilities p_0 and p_1. Now, this algorithm will generate a finite C_0 only if it generates a finite $\vec{\mathbb{Z}}^2_{p_0,p_1}$. Thus the probability of the former event, namely $1 - P_\infty$, is bounded from above by the probability that the latter event happens.

Theorem 8. *Let $0 < \delta < 1/3$. Let $p = (1+\varepsilon)/2n$, where $\varepsilon = \varepsilon(n) = 2n^{\delta-1/3}$. Then the probability that $0 \in \mathbb{Z}^{n+2}$ belongs to $\mathbb{Z}_p^{n+2} \in \mathscr{G}_{\text{ind}}(\mathbb{Z}^{n+2}, p)$ and lies in an infinite component is at least $(2+o(1))\varepsilon p > 0$.*

Proof. Let us write

$$K_i = \{(x_1, x_2)\colon\ x_1, x_2 \geqslant 0 \text{ and } x_1 + x_2 = i\}$$

and

$$L_j = \{(x_i)_1^{n+2}\colon\ (x_1, x_2) \in K_j\} = \bigcup_{(a,b)\in K_j} H_{a,b},$$

where the $H_{a,b}$ are the n-dimensional flats defined in the previous section.

To prove our result it suffices to show that, conditional on the event that $0 \in \mathbb{Z}^{n+2}$ belongs to $\mathbb{Z}_p^{n+2}$, the probability that $0 \in \mathbb{Z}^{n+2}$ lies in an infinite component is at least $(2+o(1))\varepsilon$. To analyse this conditional probability, we consider Algorithm III, given in Figure 3.

begin
  pick a random $H^p_{0,0} \in \mathscr{G}_{\text{ind}}(H_{0,0}, p)$;
  add 0 into $H^p_{0,0}$;
  let C be the component of $0 \in \mathbb{Z}^{n+2}$ in $H^p_{0,0}$;
  $W := \varnothing$;
  if $|C| < n^{10}$ **then** halt
  else begin
    $A := V(C)$; $W := \{(0,0)\}$; $i := 0$;
    repeat
      $i := i + 1$;
      for every $(a, b) \in K_i$ **do**
        $S_{a,b} := \{x \in H_{a,b}\colon x$ is adjacent to $A\}$;
      $A := \varnothing$;
      for every $(a, b) \in K_i$ **do begin**
        pick a random $H^p_{a,b} \in \mathscr{G}_{\text{ind}}(H_{a,b}, p)$;
        if for some $x \in S_{a,b}$ such that $x \in H^p_{a,b}$ the
          component C_x of x in $H^p_{a,b}$ is such that $|C_x| \geqslant n^{10}$
        then begin {*success*}
          $A := A \cup V(C_x)$;
          $W := W \cup \{(a, b)\}$
          end {*success*}
      end
    until $A = \varnothing$
    end
end

Figure 3. Algorithm III.

We claim that

$$\mathbb{P}(|C_0| = \infty \mid 0 \in \mathbb{Z}_p^{n+2}) \geqslant \mathbb{P}(\text{Algorithm III does not halt}).$$

Indeed, if a vertex $(a, b) \in \mathbb{Z}_+^2$ belongs to W then there is a vertex in $H_{a,b}$ that is connected to $0 \in \mathbb{Z}^{n+2}$. Now, note that every pair (a, b) has only one chance of being put into W. Thus if the algorithm does not halt then infinitely many pairs in $\mathbb{Z}_+^2$ will belong to W, since in every execution of the **repeat** loop either W is increased or A ends up empty. Thus the claim follows.

Let us now estimate from below the probability that Algorithm III does not halt. Note that Algorithm III, by keeping certain vertices of $\mathbb{Z}_+^2$ in W, simulates the generation of $\vec{\mathbb{Z}}_{p_0,p_1}^2$ for certain p_0 and p_1. Indeed, at every stage the set $W \subset \mathbb{Z}_+^2$ in Algorithm III keeps the vertices of $\mathbb{Z}_+^2$ known to belong to $\vec{\mathbb{Z}}_{p_0,p_1}^2$, where p_0 is the probability that the origin $0 \in H_{0,0}$ belongs to a component of order at least n^{10} in $H_{0,0}^p$, conditional on the event that $0 \in H_{0,0}^p$, and p_1 is the probability that we go through the block marked *success*. Thus $p_0 = (2 + o(1))\varepsilon$ by Corollary 6, and $p_1 = 1 - o(1)$ by Lemma 7. Now note that Algorithm III halts only when $\vec{\mathbb{Z}}_{p_0,p_1}^2$ generated in this way is finite. Thus

$$\mathbb{P}(\text{Algorithm III does not halt}) = \mathbb{P}(|\vec{\mathbb{Z}}_{p_0,p_1}^2| = \infty) \geqslant (2 + o(1))\varepsilon,$$

as required. □

Combining Theorem 8 with some well-known results about critical probabilities, we obtain the following:

THEOREM 9. (i) *We have*

$$\frac{1}{2n-1} \leqslant p_c(\mathbb{Z}^n, \text{bond}) \leqslant p_c(\mathbb{Z}^n, \text{site}) \leqslant \frac{1 + n^{o(1)-1/3}}{2n}. \tag{4}$$

(ii) *There is a function $\delta = \delta(n) = o(1)$ such that if $\varepsilon = \varepsilon(n) \geqslant n^{\delta - 1/3}$ and $p = (1+\varepsilon)/2n$, then $\theta_{\text{ind}}^n(p) = \mathbb{P}(|C_0| = \infty \mid 0 \in \mathbb{Z}_p^n) = (1 + o(1))\pi_{2n}^-(p)$.*

PROOF. (i) The first inequality in (4) is due to Broadbent and Hammersley (see [6, 7]). The second inequality has been discovered several times; see McDiarmid [16], Hammersley [13] and Oxley and Welsh [17]. The third and last inequality follows from Theorem 8.

(ii) This is a simple reformulation of Theorem 8. □

6. BOND PERCOLATION

An immediate corollary of Theorem 9 is that $p_c(\mathbb{Z}^n, \text{bond}) \leqslant 1/2n + O(1/n^{o(1)+4/3})$. Recently, Hara and Slade [12] proved an essentially best possible bound for this critical probability. Indeed, when investigating the validity of the so-called triangle condition in certain percolation processes in $\mathbb{Z}^n$, they showed that

$$p_c(\mathbb{Z}^n, \text{bond}) = \frac{1}{2n} + O\left(\frac{1}{n^2}\right).$$

(The triangle condition is a far-reaching property, implying, for instance, the existence of certain critical exponents and the continuity of the percolation density at the critical probability.) The basic tool used in [12] is an expansion related to the lace expansion for self-avoiding walks.

Coupled with a certain technique used in [3, Section 4], the methods used in this note can be used to give a quick proof for a slightly weaker bound for $p_c(\mathbb{Z}^n, \text{bond})$. To conclude this note, we shall sketch this proof.

The starting point is a result analogous to Corollary 4, the proof of which is based on a strengthening of Lemma 2 for large k (see [3, Lemma 8 and Corollary 20]).

Lemma 10. *Let $p=(1+\varepsilon)/2n$, where $(\log n)^2/n \leqslant \varepsilon=\varepsilon(n) \leqslant 1$. Let v be a fixed vertex of $\mathbb{Z}^n$ and write $n_0=\lceil n/20\varepsilon \rceil$. Then*

$$\mathbb{P}(|C_v| \geqslant n_0) \geqslant (1+o(1))\pi_{2n}^-(p),$$

where $C_v=C_v(\mathbb{Z}_p^n)$ is the component of $\mathbb{Z}_p^n \in \mathcal{G}(\mathbb{Z}^n, p)$ containing v.

It is well known that in a Galton–Watson branching process, the probability of having total progeny k, conditional on the process dying out, is exponentially small. From this it follows that these processes tend to have total progeny either very small or infinite. Using some standard isoperimetric inequalities, a similar assertion can be proved for the process in $\mathbb{Z}^n$ with 'infinite' replaced by 'very large'. For the rest of this section, let us fix a constant $C \geqslant 1$.

Theorem 11. *Let $p=(1+\varepsilon)/2n$, where*

$$96(3C+1)(\log n)^2 n^{-1} \leqslant \varepsilon=\varepsilon(n) \leqslant (3/80)\{(C+1)\log n\}^{-1}. \tag{5}$$

Let $v \in \mathbb{Z}^n$ be fixed and let $C_v=C_v(\mathbb{Z}_p^n)$ be the component of $v \in \mathbb{Z}^n$ in $\mathbb{Z}_p^n \in \mathcal{G}(\mathbb{Z}^n, p)$. Then $\mathbb{P}(n/20\varepsilon \leqslant |C_v| \leqslant n^{C \log n}) \leqslant 2n^{-\log n}$.

The above result implies that in the conclusion of Lemma 10 we may replace $n_0=\lceil n/20\varepsilon \rceil$ by $n^{C\log n}$, if we assume that ε satisfies (5). This strengthening of Lemma 10 may be used to prove a result analogous to Lemma 7.

Lemma 12. *Let $C \geqslant 6$ be fixed. Let $p=(1+\varepsilon)/2n$, where $\varepsilon=\varepsilon(n)=192 \times (3C+1)(\log n)^2 n^{-1}$. Let a set $S \subset H_{a,b}$ with $|S| \geqslant 2n^{(C/2)\log n}$ be given. Let $C_x=C_x(H_{a,b}^p)$ be the component of $x \in H_{a,b}$ in the random graph $H_{a,b}^p \in \mathcal{G}(H_{a,b}, p)$. Then*

$$\mathbb{P}(|C_x| \geqslant 5n^{1+(C/2)\log n} \text{ for some } x \in S)=1-o(1).$$

The proof of Lemma 12 may be obtained by slightly modifying the proof of Lemma 7. Armed with Lemma 12, we can prove our main result for bond percolation in the same way as we derived Theorem 8 from Lemma 7.

Theorem 13. *We have*

$$\frac{1}{2n-1} \leqslant p_c(\mathbb{Z}^n, \text{bond}) \leqslant \frac{1}{2n}+O\left(\frac{(\log n)^2}{n^2}\right).$$

Moreover, if $\varepsilon=\varepsilon(n) \geqslant (\log n)^2/n$ and $p=(1+\varepsilon)/2n$, then

$$\theta^n(p)=\mathbb{P}(|C_0|=\infty)=(1+o(1))\pi_{2n}^-(p).$$

It would be very interesting to close the gap between the estimates for $p_c(\mathbb{Z}^n, \text{site})$ given in (4). Our upper bound is most unlikely to be sharp and the lower bound $(2n-1)^{-1}$ is essentially trivial, and hence we believe that there is room for improvement on both sides. In particular, it would be interesting to see whether extensions of isoperimetric inequalities could be used to prove a better result than Lemma 5.

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